# NUMERICAL STUDY OF THE PHASE SPACE OF A FOUR DIMENSIONAL SYMPLECTIC MAP

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## 1. Introduction

We study the structure of non periodic orbits in a 4–D symplectic map, composed of two coupled 2–D maps. Such maps correspond to 3 degrees of freedom Hamiltonian systems.

The 4–D map is

$$\begin{array}{ll} x_1' = x_1 + x_2, & x_2' = x_2 - \nu \sin(x_1 + x_2) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]\,, \\ x_3' = x_3 + x_4, & x_4' = x_4 - \kappa \sin(x_3 + x_4) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]\,, \end{array} \quad (\mathrm{mod}\, 2\pi) \quad (1)$$

This is a variant of Froeschlé's 4–D symplectic map [1], and its periodic orbits have been studied by Contopoulos and Giorgilli [2].

## 2. Projection of the orbit on the plane $(\mathbf{x_1}, \mathbf{x_2})$

In Fig. 1a we see the projections of the successive consequents of some orbits on the plane $(x_1, x_2)$ for $\nu = 10^{-3}$, $\kappa = 10^{-1}$ and $\mu = 10^{-5}$. At the center we see the point of the stable periodic orbit $x_1 = x_2 = x_3 = x_4 = 0$. Around it the projections of the consequents are near closed invariant curves. The last orbit starting at the point $A = (3, 0, 0, 0)$ does not give a closed curve, but its successive consequents move from the right to the left of the picture forming a line. When the line reaches the left boundary it continues from the right boundary to the left, because $x_1$ is given mod $2\pi$. The successive lines move towards smaller values of $x_2$. After about 312 500 iterations the $x_2$ coordinate reaches the value $-\pi$, and the orbit continues with $x_2 = +\pi$, and again moves towards smaller values of $x_2$. After about 625 000 iterations it reaches the neighborhood of the initial point A, but the orbit is not periodic.

The consequents of the orbit are ordered in a particular way. As the orbit goes further away from the initial point it forms resonance zones (sets of parallel curves, Fig. 1b). These zones are more or less parallel to the $x_1$-axis. The successive consequents are not on successive curves in general. For example in Fig. 1b we mark a zone with 14 curves, in which the successive points are on every third curve defining the resonance $m/n = 3/14$. As we easily see the zone of the resonance 3/14 is formed in the region where $x_2 \approx -2\pi(3/14) = -1.346$. In Fig. 1b we see also many other resonances.

The structure formed from the projections of the consequents on the plane $(x_1, x_2)$, is influenced strongly by the resonances of the corresponding 2–D map $(x_1, x_2)$, $(\mu = 0)$. In particular the 2–D map $(x_1, x_2)$ for $\nu = 0$ is composed of two independent 1-dimensional maps: $x_2 =$ const., and $x_1$ is found by adding repeatedly the value of $x_2$. So when $x_2 =$



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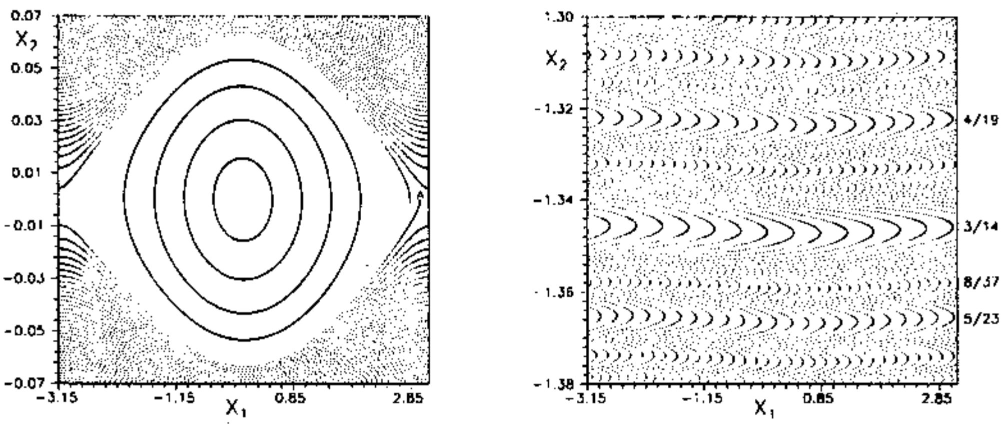


*Figure 1.* (a) Successive consequents of a non periodic orbit in the 4–D map (1), starting at the point A, on the right of the central region that is filled with invariant curves. (b) Successive consequents of this periodic orbit in a region containing the resonances $m/n = 4/19, 3/14, 8/37$ and $5/23$.

$-2\pi m/n$ (mod $2\pi$) with $m, n \in \mathbb{N}^*$, $m \leq n$ we have $n$ points which are plotted by $m$ on the plane $(x_1, x_2)$. So in the 4–D map we have a horizontal zone of the corresponding resonance when $x_2 \approx -2\pi m/n$ (mod $2\pi$), where the points form $n$ curves and the successive points are plotted on every $m$-th curve. The points of the orbit form curves of all the possible resonances which are arranged along a Farey tree.

To obtain the Farey tree we define the zeroth generation by a pair of rationals $m_1/n_1$ and $m_2/n_2$ with $m_1 n_2 - m_2 n_1 = \pm 1$. Such rationals are called "neighboring". A rational between two neighbors is obtained by adding numerators and denominators $m_3/n_3 = (m_1 + m_2)/(n_1 + n_2)$. This rational has the smallest denominator between the two neighbors, and it is a neighbor to each of its parents. By continuing this construction, we eventually obtain every rational in the interval $[m_1/n_1, m_2/n_2]$ up to some given order of the denominator.

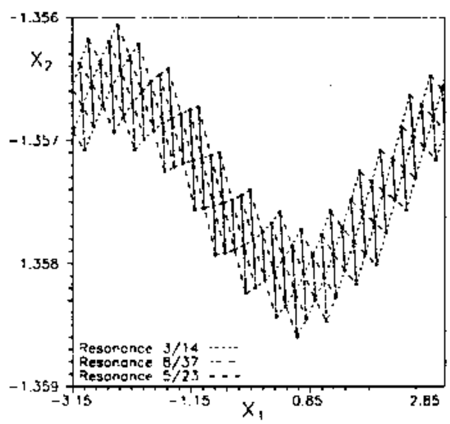


*Figure 2.* Enlargement of 100 points of the region of the resonance 8/37. The resonance is best visible in Fig. 1b because its curves are almost perpendicular to $x_1$-axis. The curves of resonances 3/14 and 5/23 are tilted.

In general we can put the points of the orbit on curves of any resonance. The curves of a particular resonance $m/n$ are very tilted except in the region $x_2 \approx -2\pi m/n$ (mod $2\pi$) where they become more or less perpendicular to the $x_1$-axis and the curves of the resonance are clearly seen. For example in Fig. 2 we see in enlargement 100 points from the region of the resonance 8/37 (Fig. 1b), and three kinds of curves that the points are forming. We see that if the points are in a region before the appearance of a resonance (i. e. above the resonance) its curves are oblique with negative slope. This happens to the curves of the resonance 5/23 in Fig. 2, because, as we see in Fig. 1b, the points at the resonance 8/37 are above the resonance 5/23. On the other hand if the points are in a region after the appearance of a resonance (i.e., below the resonance) its curves are oblique with positive slope. This happens to the curves of the resonance 3/14 in Fig. 2, because, as we see in Fig. 1b, the points at the resonance 8/37 are below the resonance 3/14. Finally, if the points are close to a resonance, the curves are almost perpendicular to the $x_1$-axis. This happens to the curves of the resonance 8/37 in Figs. 1b and 2.

Since the number of points needed to cover the plane $(x_1, x_2)$ is finite (about 625 000 in our case) and the rational numbers are infinite we understand that on the plane $(x_1, x_2)$ only resonances up to some order are formed, which means up to some maximum value of the denominator.

## 3. The width of the zones of the resonances on the plane $(\mathbf{x}_1, \mathbf{x}_2)$

In Fig. 1b it is clearly seen that the width of the zones of the resonances varies. For example the zone of the resonance 4/19 is smaller than the zone of the resonance 3/14. We empirically measure the width of a zone along the $x_2$-axis, by defining its limits when its curves become more or less parallel to the $x_1$-axis. When we have resonances of high order, which means large denominators, the width is very small and even if we enlarge its region, we cannot determine it easily. So we measure the width of resonances which are clearly seen and have well defined limits. Another difficulty is that the limits of the resonance 1/1 around the main island for $x_2 \approx 0$ (Fig. 1a) are not clear. So we find the width of resonances with denominators $\geq 2$.

For $\nu = 10^{-3}$, $\kappa = 10^{-1}$ and $\mu = 10^{-5}$ we measure the width of the zones of resonances with denominator starting from 2 up to 20. In Fig. 3a we plot the width $(W)$ of the zones of various resonances as a function of $m/n$. We see that there is a structure, e.g., a symmetry with respect to the line $m/n = 1/2$. It is clear that the widths of high order resonances are smaller than the ones with small denominators. By line connecting the points of resonances with the same denominator we see that their width is almost constant, i.e., the width of a resonance zone does not depend on the numerator of the resonance.

In Fig. 3b we plot the width $(W)$ of each resonance as a function of the denominator $n$ and we find a best fitting line of the form

$$W = A/n^B, \qquad \text{where} \qquad A = 0.190896\,, \quad B = 1.04929\,. \tag{2}$$

With some tolerance we can say that the width of the zone of a resonance is inversely proportional to its denominator. This law explains the structure we see in Fig. 3a.

Since the sum $\sum_1^\infty 1/n$ is infinite while the total interval of $x_2$ is finite $(2\pi)$, we understand that we can only see resonances up to some order, which means up to some maximum value of the denominator. By looking carefully at the projections of the consequents on the plane $(x_1, x_2)$, we found empirically that the maximum order $n_{0,\text{emp}}$ of clearly formed resonances is 75 (we can see the resonance 7/75). Thus $n_{0,\text{emp}} \approx 75$.

Since the resonance $m/n$ is visible when $x_2 \approx -2\pi m/n \pmod{2\pi}$, and its width is given from (2) for $B = 1$, we consider that its zone is approximately the interval

$$x_2 \in [-2\pi m/n - W/2, -2\pi m/n + W/2] \pmod{2\pi}\,. \tag{3}$$

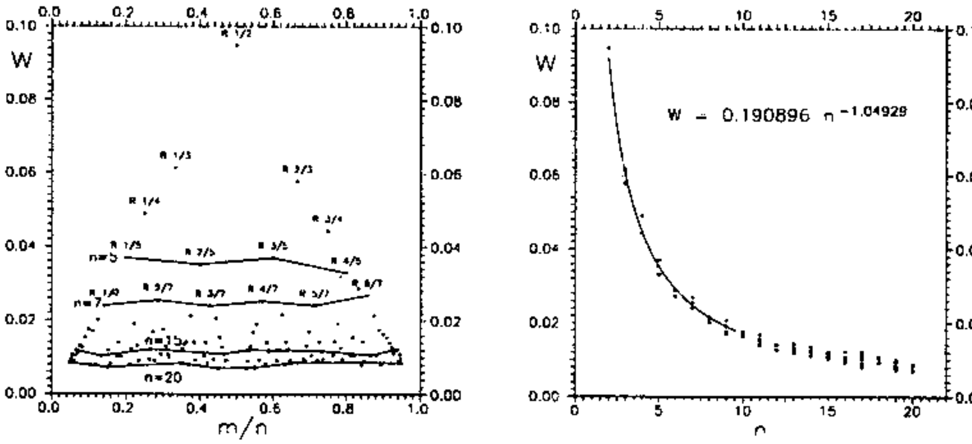


*Figure 3.* (a) The width $W$ of the zones of resonances with denominators from 2 up to 20, as a function of $m/n$. The widths of some resonances are marked. The points of resonances with denominators 5, 7, 15 and 20 are line connected. (b) The width $W$ of the zones of resonances $m/n$ as a function of the denominator $n$. For every $n$ we have a number of points which correspond to the various values of the numerator $m$. The best fitting line $W = 0.190896\, n^{-1.04929}$ is also plotted.

Thus in order to find how many resonances up to a certain denominator are needed to cover the segment $x_2 \in [-\pi, \pi)$, we make a grid of step 0.0001 of the values of $x_2$ and we find the maximum denominator $n_0$ needed in order to put all the values of the grid at least in one segment of the form (3). We find that $n_0 \approx 65$. This value is not far from the empirical limit $n_{0,\text{emp}}$. Thus the assumption that the width of the zone of a resonance is inversely proportional to its denominator is valid for all the resonances for $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 10^{-5}$.

We do the same study of the widths for larger values of the coupling parameter $\mu$. For $\nu = 10^{-3}$, $\kappa = 10^{-1}$ and $\mu = 5\,10^{-5}$ the orbit with initial conditions $x_1 = 3$, $x_2 = x_3 = x_4 = 0$, produces a similar structure on the plane $(x_1, x_2)$. It needs about 126 000 iterations to fill the plane $(x_1, x_2)$ instead of 625 000 for $\mu = 10^{-5}$. By measuring the width of the zones of resonances with denominator from 2 to 15 we get the same law (2) for the width, with:

$$A = 0.311524\,, \qquad B = 1.0735\,. \tag{4}$$

Again $B \approx 1$. We also find $n_{0.\text{emp}} \approx 46$, $n_0 \approx 40$.

For $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 10^{-3}$ the same structure remains. The orbit with initial conditions $x_1 = 3$, $x_2 = x_3 = x_4 = 0$, needs only 6300 iterations to fill the plane $(x_1, x_2)$. By measuring the width of the zones of resonances with denominator from 2 to 13 we get the same law (2) for the width, with:

$$A = 0.678511\,, \qquad B = 0.980165\,. \tag{5}$$

Again $B \approx 1$, and we also find $n_{0,\text{emp}} \approx 18$, $n_0 \approx 18$.

Thus the width of the zone of a resonance is always inversely proportional to its denominator and it grows as $\mu$ increases because the value of $A$ increases as we can see from (2), (4) and (5). This increase is given approximately by a power law: $A = 4.53275\,\mu^{0.273411}$. Thus in general we get

$$W\,(\mu, m/n) \approx 4.53\,\mu^{0.27}/n\,. \tag{6}$$

The structure on the plane $(x_1, x_2)$ and the law (6) are valid for $\mu$ up to order $10^{-1}$, because for larger values of $\mu$ we have a small number of points on the plane $(x_1, x_2)$ and we cannot see the zones of resonances formed.

## 4. Conclusions

We studied the phase space of a 4–D map composed of two coupled 2–D maps, in order to understand how the features of the two 2–D maps influence the 4–D map, as we change the coupling parameter $\mu$. We investigated a class of non periodic orbits, starting outside the main island around $(x_1 = 0, x_2 = 0)$, of the limiting 2-dimensional case $\mu = 0$, for various values of $\mu$.

We studied in detail how the orbits evolve in time. On the plane $(x_1, x_2)$ the various resonances $m/n$ influence parts of the orbit in certain zones almost parallel to the $x_1$-axis when $x_2 \approx -2\pi m/n \pmod{2\pi}$. The width of these zones is given by the empirical law (6).

The structure we described in section 2 and the empirical law (6) are general features of the 4–D map, for small values of $\mu$ ($\mu < 10^{-1}$). They are valid for orbits with initial conditions on the plane $(x_1, x_2)$ outside the main island, and everywhere on the plane $(x_3, x_4)$.

## Acknowledgements

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## References

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